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REGULARITY OF NONLINEAR FLOWS ON MANIFOLDS: NONLINEAR ESTIMATES ON NEW TYPE VARIATION OR WHY GENERALIZATIONS OF CLASSICAL COVARIANT DERIVATIVES ARE REQUIRED?

The correct approach to the regularity parabolic problems on manifolds requires the study of dependence between the coefficients of equation and geometric characteristics of manifold, like curvature.

We demonstrate that the geometrically correct work with the nonlinear differential flow on manifold leads to the introduction of a new type variations with respect to the initial data. They are defined via a natural generalization of covariant Riemannian derivative to the case of diffeomorphisms.

Using these variations we find how the curvature manifests in the structure of high order variational equations and determine a set of a priori nonlinear estimates on any order regularity. In particular, we derive the regular properties of corresponding solutions to parabolic equations on non-compact manifolds with the growing on infinity coefficients.

This paper develops results of [2]-[7] to the manifold case.

1. Invariant representations of semigroups derivatives: statement of problem.

For simplicity, let us consider a second order parabolic equation

$$\frac{\partial}{\partial t}u(t,x) = Lu(t,x), \quad u(0,x) = f(x) \tag{1}$$

on noncompact connected oriented smooth Riemannian manifold M without boundary. The second order differential operator

$$Lf = A_0 f + \frac{1}{2}\sum_\alpha A_\alpha(A_\alpha f). \tag{2}$$

is expressed in terms of smooth vector fields $A_0,\ A_\alpha,\ \alpha = 1,...,d,\ d = \dim M$, globally defined on M.

Equation (1) is related with Stratonovich diffusion

$$y_t^x = x + \int_0^t A_0(y_s^x)ds + \sum_{\alpha=1}^d \int_0^t A_\alpha(y_s^x)\delta W_s^\alpha, \quad x \in M \tag{3}$$

via Kolmogorov representation of corresponding diffusion semigroup

$$u(t,x) = (P_t f)(x) = \mathbb{E}f(y_t^x) \tag{4}$$

Traditionally equation (3) is understood in a sense that for any smooth function with compact support $f \in C_0^2(M)$ the following equation

$$f(y_t^x) = f(x) + \int_0^t (A_0 f)(y_s^x)ds + + \sum_{\alpha=1}^d \int_0^t (A_\alpha f)(y_s^x)\delta W_s^\alpha \tag{5}$$

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holds as usual equation in $I\!R^1$, W^α_s denotes the standard $I\!R^d$-valued Wiener process. In particular, one can take functions $f^i(x) = x^i$ to be local coordinates to find the consistent family of local stochastic equations.

In this talk we are going to find how the properties of nonlinear diffusion A_α and drift A_0 coefficients should be related with the geometric properties of manifold to lead to the smooth properties of diffusion semigroup (4) and to the regular dependence of process y^x_t on initial data. We remark, that henceforth we use the uncommon notation y^x_t for the diffusion process instead of traditional Greek letters, because in this article we develop purely nonstochastic methods, that are still valid for the case of ordinary differential equations (i.e. when diffusion coefficients $A_\alpha \equiv 0$).

By Kolmogorov representation (4) we have to find the geometrically invariant representations for semigroups derivatives in terms of some invariant (and still not introduced) derivatives of process y^x_t with respect to the initial data.

Taking formally the first order derivative of (4) we find

$$\nabla_k P_t f(x) = \frac{\partial}{\partial x^k} \mathbb{E} f(y^x_t) = \mathbb{E} \frac{\partial f(y^x_t)}{\partial y^m} \frac{\partial (y^x_t)^m}{\partial x^k} = \mathbb{E} \nabla^y_m f(y^x_t) \frac{\partial (y^x_t)^m}{\partial x^k} \tag{6}$$

Due to the properties of Jacobians, the above representation is invariant with respect to the local coordinate transformations $(x) \to (x')$. Moreover, the similar arguments demonstrate that the first order variation $\dfrac{\partial (y^x_t)^m}{\partial x^k}$ of diffusion with respect to the initial data represents

- covector field on index k with respect to coordinate transformations in the vicinity of initial data $(x) \to (x')$
- vector field on index m with respect to the choice of local coordinate vicinity for diffusion $(y) \to (y')$.

Representation (6) relates the first order covariant derivatives of function with the covariant derivatives of its evolution, i.e. provides natural tools for the study of the first order regularity problems.

Turning to the higher order differentiability of diffusion semigroup P_t we have to *define the higher order variations* of process y^x_t. The attempt to write classical covariant derivatives of the first order variation

$$\nabla^x_{k_n} \dots \nabla^x_{k_2} \frac{\partial (y^x_t)^m}{\partial x^{k_1}},$$

would be naive. Though they are invariant with respect to the local coordinates transformations $(x) \to (x')$, the property, that first order variation is a vector field on index m with respect to transformations $(y) \to (y')$, will be destroyed. There will inevitably arise derivatives on variable x of Jacobians $\dfrac{\partial (y'(x,t))^n}{\partial (y(x,t))^m}$ of coordinate changes $(y) \to (y')$. Recall that the covariant derivative of a tensor field is defined in a standard way

$$\nabla^x_k u^{i_1,\dots,i_p}_{j_1,\dots,j_q} = \frac{\partial}{\partial x^k} u^{i_1,\dots,i_p}_{j_1,\dots,j_q} + \sum_{s=1}^{p} \Gamma_k{}^{i_s}{}_\ell(x) u^{i_1,\dots,i_p|_{i_s=\ell}}_{j_1,\dots,j_q} - \sum_{s=1}^{q} \Gamma_k{}^\ell{}_{j_s}(x) u^{i_1,\dots,i_p}_{j_1,\dots,j_q|_{j_s=\ell}} \tag{7}$$

$u^{i_1,\dots,i_p|_{i_s=\ell}}_{j_1,\dots,j_q}$ means substitution of index i_s by ℓ, the summation on repeating indexes is implemented, and $\Gamma(x)$ denote the connection coefficients.

To find the high order representations of semigroup derivatives we may directly write the second order covariant derivative of semigroup

$$\nabla_k\nabla_j P_t f(x) = \{\frac{\partial}{\partial x^k}\frac{\partial}{\partial x^j} - \Gamma_{k\ j}^{\ h}(x)\frac{\partial}{\partial x^h}\}P_t f(x) = \mathbb{E}\{\frac{\partial}{\partial x^k}\frac{\partial}{\partial x^j} - \Gamma_{k\ j}^{\ h}(x)\frac{\partial}{\partial x^h}\}f(y_t^x) =$$

$$= \mathbb{E}\{\frac{\partial f(y)}{\partial y^m}\frac{\partial^2 y^m}{\partial x^k \partial x^j} + \frac{\partial^2 f(y)}{\partial y^m \partial y^n}\frac{\partial y^m}{\partial x^k}\frac{\partial y^n}{\partial y^j} - \Gamma_{k\ j}^{\ h}(x)\frac{\partial f(y)}{\partial y^m}\frac{\partial y^m}{\partial x^h}\}$$

and form the covariant derivatives of f in the r.h.s., using that $\nabla_\ell^y f(y) = \frac{\partial}{\partial y^\ell}f(y)$ and $\nabla_m^y\nabla_n^y f(y) = \frac{\partial}{\partial y^m}\frac{\partial}{\partial y^n}f(y) - \Gamma_{m\ n}^{\ \ell}\frac{\partial}{\partial y^\ell}f(y)$. We find

$$\nabla_k\nabla_j P_t f(x) = \mathbb{E}\{\,(\nabla_m^y\nabla_n^y f(y) + \Gamma_{m\ n}^{\ \ell}(y)\nabla_\ell^y f(y)\,)\frac{\partial y^m}{\partial x^k}\frac{\partial y^m}{\partial x^j} + \tag{8}$$

$$+\nabla_m^y f(y)(\frac{\partial^2 y^m}{\partial x^k \partial x^j} - \Gamma_{k\ j}^{\ h}(x)\frac{\partial y^m}{\partial x^h})\,\} =$$

$$= \mathbb{E}\{\nabla_m^y\nabla_n^y f(y)\frac{\partial y^m}{\partial x^k}\frac{\partial y^n}{\partial x^j} + \nabla_m^y f(y)(\frac{\partial^2 y^m}{\partial x^k \partial x^j} - \Gamma_{k\ j}^{\ h}(x)\frac{\partial y^m}{\partial x^h} + \Gamma_{\ell\ n}^{\ m}(y)\frac{\partial y^\ell}{\partial x^k}\frac{\partial y^n}{\partial x^j}\,)\,\} \tag{9}$$

The first term $\nabla^y\nabla^y f\cdot\frac{\partial y}{\partial x}\frac{\partial y}{\partial x}$ is obviously invariant under transformations $(x)\to(x')$ and $(y)\to(y')$. However arises a problem of terms in brackets

$$\frac{\partial^2 y^m}{\partial x^k \partial x^j} - \Gamma_{k\ j}^{\ h}(x)\frac{\partial y^m}{\partial x^h} + \Gamma_{\ell\ n}^{\ m}(y)\frac{\partial y^\ell}{\partial x^k}\frac{\partial y^n}{\partial x^j}\ ? \tag{10}$$

Traditional approach to treat these terms is to form the covariant derivative on x variable form first and second terms

$$(10) = |\ 1^{st}\ way\ |\ = \nabla_k^x(\frac{\partial y^m}{\partial x^j}) + \Gamma_{\ell\ h}^{\ m}(y)\frac{\partial y^\ell}{\partial x^k}\frac{\partial y^n}{\partial x^j} \tag{11}$$

Such representation is obviously invariant with respect to transformations $(x)\to(x')$.

We remark that the third term with connection $\Gamma(y)$ has transformation of coordinates law, that includes the second order derivatives of coordinate change, similar to Ito formula, and seems to compensate the influence of stochastics. However, our considerations also hold for the case of ordinary differential equations ($A_\alpha \equiv 0$). Therefore in further research we do not follow the stochastic arguments.

It is not clear, whether representation (11) is invariant with respect to the coordinate changes $(y)\to(y')$ for process y_t^x. Let us work with (10) in other way, by collecting first and third terms together

$$(10) = |2^{nd}\ way| = \frac{\partial}{\partial x^k}(\frac{\partial y^m}{\partial x^j}) - \Gamma_{k\ j}^{\ h}(x)\frac{\partial y^m}{\partial x^h} + \Gamma_{\ell\ h}^{\ m}(y)\frac{\partial y^\ell}{\partial x^k}\frac{\partial y^n}{\partial x^j} =$$

$$= \frac{\partial y^\ell}{\partial x^k}\frac{\partial}{\partial y^\ell}(\frac{\partial y^m}{\partial x^j}) - \Gamma_{k\ j}^{\ h}(x)\frac{\partial y^m}{\partial x^h} + \Gamma_{\ell\ h}^{\ m}(y)\frac{\partial y^\ell}{\partial x^k}\frac{\partial y^n}{\partial x^j} =$$

$$= \frac{\partial y^\ell}{\partial x^k}\nabla_\ell^y(\frac{\partial y^m}{\partial x^j}) - \Gamma_{k\ j}^{\ h}(x)\frac{\partial y^m}{\partial x^h} \tag{12}$$

where we used that $\dfrac{\partial}{\partial x^k}=\dfrac{\partial y^\ell}{\partial x^k}\dfrac{\partial}{\partial y^\ell}$. This representation, in comparison to (11), is invariant with respect to the coordinate changes $(y)\to(y')$.

We come to the conclusion that all terms in (10) define a second variation of process y_t^x and represent

- vector field on index m with respect to the "Ito"changes of coordinates $(y)\to(y')$
- twice covariant field on indexes k,j with respect to the "diff.geometric"changes of coordinates $(x)\to(x')$

Because the arguments above also work for the choice $A_\alpha\equiv 0$, i.e. in the ordinary differential equations case, when no stochastics appear at all, the introduction of high order variation is a pure question of differential geometry.

If one knows how to define the second order variation then its high order analogies could be easily written.

Definition 1. *High order variations $\mathbb{V}_\gamma^x y_t^x$, $\gamma=\{k_1,...,k_n\}$, of process y_t^x are defined by recurrent relations*

$$\mathbb{V}_k^x y^m=\frac{\partial(y_t^x)^m}{\partial x^k},$$

$$\mathbb{V}_k^x(\mathbb{V}_\gamma^x y^m)=\nabla_k^x(\mathbb{V}_\gamma^x y^m)+\Gamma_{p\ q}^{\ m}(y_t^x)\mathbb{V}_\gamma^x y^p\frac{\partial y^q}{\partial x^k}= \tag{13}$$

$$=\partial_k^x(\mathbb{V}_\gamma^x y^m)-\sum_{j\in\gamma}\Gamma_{k\ j}^{\ h}(x)\mathbb{V}_{\gamma|_{j=h}}^x y^m+\Gamma_{p\ q}^{\ m}(y_t^x)\mathbb{V}_\gamma^x y^p\frac{\partial y^q}{\partial x^k}$$

The last term with $\Gamma(y)$ in (13) depends on solution y_t^x and therefore generalizes the classical Riemannian covariant derivative. The invariance of (13) with respect to $(x)\to(x')$ transformations is obvious, for transformations in image $(y)\to(y')$ one should argue like in (12), e.g. [7].

Using variations $\mathbb{V}_\gamma^x y_t^x$, we can now write invariant representations of semigroup's derivatives:

Theorem 2. *The covariant derivatives of function and its evolution are related via new type variations by*

$$\nabla_\gamma^x P_t f(x)=\sum_{\delta_1\cup\ldots\cup\delta_s=\gamma}\mathbb{E}\,(\nabla^y_{\{j_1,\ldots,j_s\}}f)(y_t^x)\,\mathbb{V}_{\delta_1}^x y^{j_1}\ldots\mathbb{V}_{\delta_s}^x y^{j_s} \tag{14}$$

Here $\nabla_\gamma^x=\nabla_{k_1}^x\ldots\nabla_{k_n}^x$ for $\gamma=\{k_1,...,k_n\}$.

Proof. This representation is easily verified recurrently. Indeed, suppose it is true for all $|\gamma|\le n$, then, similar to (8), one should consider

$$\nabla_k^x\nabla_\gamma^x P_t f(x)=\partial_k^x\nabla_\gamma^x P_t f(x)-\sum_{j\in\gamma}\Gamma_{k\ j}^{\ h}(x)\nabla_{\gamma|_{j=h}}^x P_t f(x),$$

substitute here expressions (14), add and subtract $\Gamma(y)$ to form the high order covariant derivatives of f, redenote summation indices and come to the representation (14) for $\nabla_k^x\nabla_\gamma^x P_t f$. $\square$

2. Recurrent form of the high order variational equations.

Further step will be to find the equation on high order variations. Differentiating (3) on initial data x we have

$$\delta(\frac{\partial y^m}{\partial x^k}) = (\frac{\partial}{\partial x^k} A^m_\alpha(y))\delta W^\alpha + (\frac{\partial}{\partial x^k} A^m_0(y))dt \tag{15}$$

The problem is to find recurrent representation for high order variational equations.

To proceed further we need some generalization of Definition 1 from process y^x_t to tensors of (x) and (y^x_t) coordinates.

DEFINITION 3. *Object $u^{(i/\alpha)}_{(j/\beta)}$ forms a mixed tensor with respect to the coordinate changes $(x) \to (x')$ and $(\phi) \to (\phi')$ iff its coordinates*

$$u^{(i/\alpha)}_{(j/\beta)} = u^{i_1 \dots i_p/\alpha_1 \dots \alpha_r}_{j_1 \dots j_q/\beta_1 \dots \beta_s}$$

form $T^{p,q}_x M$ tensor on multiindexes $(i) = (i_1, ..., i_p)$, $(j) = (j_1, ..., j_q)$ with respect to the local coordinates (x^k) and form $T^{r,s}M$ tensor on multiindexes $(\alpha), (\beta)$ with respect to the local coordinates (ϕ^m).

Now let us suppose that the new (ϕ^m) coordinates of the mixed tensor depend in effective way on the coordinates (x^k). A simple example was already provided before by stochastic flow $x \to y^x_t$, when the first order variation $\dfrac{\partial y^m(x,t)}{\partial x^k}$ was a vector on index m with respect to the coordinates $y^m(x,t)$ of process y^x_t and covector on index k in coordinate vicinity (x). Another object of this type is superposition $u^{(i)}_{(j)}(y^x_t)$ - the change of coordinates at x does not influence its values, but the choice of coordinate vicinity for y evokes the tensorial transformation law.

An analogue of Definition 1 for mixed tensors provides

DEFINITION 4. *Covariant (x)-derivative of the mixed tensor is defined by*

$$\mathbb{V}^x_k u^{(i/\alpha)}_{(j/\beta)} = \frac{\partial}{\partial x^k} u^{(i/\alpha)}_{(j/\beta)} + \sum_{s \in (i)} \Gamma_k{}^s{}_h(x) u^{(i/\alpha)|_{s=h}}_{(j/\beta)} - \sum_{s \in (j)} \Gamma_k{}^h{}_s(x) u^{(i/\alpha)}_{(j/\beta)|_{s=h}} + \tag{16}$$

$$+ \sum_{\rho \in (\alpha)} \Gamma_\gamma{}^\rho{}_\delta(\phi(x)) u^{(i/\alpha)|_{\rho=\delta}}_{(j/\beta)} \frac{\partial \phi^\delta}{\partial x^k} - \sum_{\rho \in (\beta)} \Gamma_\rho{}^\gamma{}_\delta(\phi(x)) u^{(i/\alpha)}_{(j/\beta)|_{\rho=\gamma}} \frac{\partial \phi^\delta}{\partial x^k} \tag{17}$$

Line (16) corresponds to the covariant derivative on (x^k) coordinates, additional line (17) makes the resulting expression to be tensor with respect to the coordinates (ϕ^m). One may also note that the connection symbols above depend on different parameters and the additional Jacobians $\dfrac{\partial \phi}{\partial x}$ are required in line (17).

The tensorial character of covariant (x)-derivative is easily checked, like before.

THEOREM 5. *Covariant (x)-derivative defines a tensor of higher valence, i.e. the mixed tensor law holds*

$$\mathbb{V}^x_k u^{(i/\alpha)}_{(j/\beta)} = \frac{\partial x^{k'}}{\partial x^k} \frac{\partial x^{(i)}}{\partial x^{(i')}} \frac{\partial x^{(j')}}{\partial x^{(j)}} \frac{\partial \phi^{(\alpha)}}{\partial \phi^{(\alpha')}} \frac{\partial \phi^{(\beta')}}{\partial \phi^{(\beta)}} \mathbb{V}^x_{k'} u^{(i'/\alpha')}_{(j'/\beta')}$$

Proof is an easy application of the transformation law of connection, e.g. [7].

An important property of covariant (x)-derivative is that for the superposition one has chain rule

$$\mathbb{V}^x_k u^{(\alpha)}_{(\beta)}(\phi(x)) = (\nabla_\ell u^{(\alpha)}_{(\beta)})(\phi(x))\frac{\partial \phi^\ell}{\partial x^k} \tag{18}$$

for tensor $u^{(\alpha)}_{(\beta)}$ on manifold M. Proof of this fact follows from the very definitions.

After we developed a concept of mixed tensor and its covariant (x)-derivative, we can further transform equation on the first variation (15). By adding and subtracting the terms with $\Gamma(y)$ to single out the covariant (x)-derivative of vector fields $A_0(y)$, $A_\alpha(y)$ on image coordinates (y) we have from (15)

$$\delta(\frac{\partial y^m}{\partial x^k}) = (\mathbb{V}^x_k A^m_\alpha(y) - \Gamma_p{}^m{}_q(y)A^p_\alpha\frac{\partial y^q}{\partial x^k})\delta W^\alpha + (\mathbb{V}^x_k A^m_0(y) - \Gamma_p{}^m{}_q(y)A^p_0\frac{\partial y^q}{\partial x^k})dt$$

Noting that the terms near connection contain the differential of process y we finally get equation on first variation

$$\delta(\frac{\partial y^m}{\partial x^k}) = -\Gamma_p{}^m{}_q(y)\frac{\partial y^p}{\partial x^k}\delta y^q + \mathbb{V}^x_k(A^m_\alpha(y))\delta W^\alpha + \mathbb{V}^x_k(A^m_0(y))dt \tag{19}$$

Arose an additional geometric interpretation in favor of new type variations: up to the parallel transition term with $\Gamma(y)$ the increments of first order variation are determined by covariant (x)-derivatives of coefficients. We take this observation as the recurrence base in the search for high order variational equations.

THEOREM 6. *Suppose that the equation on covariant (x)-variation $\mathbb{V}^x_\gamma y^m$, $|\gamma| \geq 1$ is written in form*

$$\delta(\mathbb{V}^x_\gamma y^m) = -\Gamma_p{}^m{}_q(\mathbb{V}^x_\gamma y^p)\delta y^q + M_\gamma{}^m{}_i\delta W^i + N^m_\gamma dt \tag{20}$$

Then the next order variation $\mathbb{V}^x_k\mathbb{V}^x_\gamma y^m = \mathbb{V}^x_{\gamma\cup\{k\}}y^m$ fulfills relation

$$\delta(\mathbb{V}^x_{\gamma\cup\{k\}}y^m) = -\Gamma_p{}^m{}_q(\mathbb{V}^x_{\gamma\cup\{k\}}y^p)\delta y^q + R_p{}^m{}_{\ell q}(\mathbb{V}^x_\gamma y^p)\frac{\partial y^\ell}{\partial x^k}\delta y^q + \tag{21}$$

$$+(\mathbb{V}^x_k M_\gamma{}^m{}_i)\delta W^i + (\mathbb{V}^x_k N^m_\gamma)dt$$

Here R forms (1,3) curvature tensor with components

$$R_1{}^2{}_{34} = \frac{\partial\Gamma_1{}^2{}_3}{\partial x^4} - \frac{\partial\Gamma_1{}^2{}_4}{\partial x^3} + \Gamma_1{}^j{}_3\Gamma_j{}^2{}_4 - \Gamma_1{}^j{}_4\Gamma_j{}^2{}_3 \tag{22}$$

where for simplicity we only point the positions of corresponding indexes.

REMARK 7. Up-to-date there is formed a set of qualitively different approaches to the construction and study of diffusion equations on manifolds. The main attention was to make *consistent the geometrical structures of manifold with the second order differentials, that arise in Itô formula for coordinate changes.*

Not complete list of known approaches includes, in particular the purely stochastic, that are based on the definition of diffusion in a consistent with geometry way by implementation of Stratonovich integrals [16, 17, 18] or more complicate description of diffusion via Itô equations in local coordinates [10, 13]. In the second case arise special Itô bundles of nontensorial fields, related with diffusion coefficients, and, to make the picture consistent, a special attention should be devoted to the normal charts, generated by exponential mappings.

Other, more geometric, approaches are related, for example, with the raise of diffusion from manifold M to the orthoframe bundle $O(M)$ over it, e.g. [11, 21]; with the consideration of manifold as embedded into $I\!R^d$ of higher dimension, e.g. [15, 24]; with interesting interpretation of Itô differential and diffusion equations as defined on the bundle of second order differential operators [25, 14]; putting forward Itô developments of equations via parallel transitions of orthoframes [12]; making more stress on properties of associated transitional probabilities [26], etc. Of course, one can also mention different peculiarities, related with many other infinite-dimensional models, e.g. [9, 15].

The procedure of making the correct correspondence between geometry and stochastics was successful in all cases. However, further question of consistency with the problematic of *differential geometry*, namely:

how the geometrically invariant differentials are constructed from invariant objects

remained in shadow. The attempts to consider the traditional derivatives in directions of vector fields or more advanced covariant and stochastic derivatives, e.g. [10, 11, 13, 21], missed an important property of geometric invariance with respect to the diffusion process y_t^x and inevitably led to the growing number of noninvariant terms in the corresponding equations. Therefore it was principally hard to trace the influence of curvature in regular properties.

Above we provided a geometrically invariant definition of new type variation $\mathbb{V}_\gamma y$. The statement of the present Theorem is in favor of this definition, namely we see that the additional term with $\Gamma(y)$ in the Definition 4 of the new invariant derivative compactificates these non-invariant terms to the *compact expressions* with curvature. So it becomes possible to find the *influence of curvature and nonlinearities of diffusion equation* on the any order regularity properties.

Finally, let us remark that the more general discussion of the invariance of differentiation operations with respect to the process y_t^x one may find in [5], where it is applied to the problem of stochastic regularity.

Proof. For simplicity we omit, where possible the dependence of connection Γ on variable y, however the dependence on x is always displayed precisely.

At first step we simply substitute the definition of covariant (x)-derivative under Stratonovich integral

$$\int \delta(\mathbb{V}_k^x \mathbb{V}_\gamma^x y^m) = \int \delta\{\,\partial_k^x \mathbb{V}_\gamma^x y^m + \Gamma_{p\ q}^{\ m}(y)\frac{\partial y^p}{\partial x^k}\mathbb{V}_\gamma^x y^q - \sum_{s\in\gamma}\Gamma_{k\ s}^{\ h}(x)\mathbb{V}_{\gamma|_{s=h}}^x y^m\} \tag{23}$$

For the first term in (23) we substitute the inductive assumption (20) and, after differentiation of integral, obtain

$$(23)_1 = \int \delta(\partial_k^x \int \{-\Gamma_{p\ q}^{\ m}(\mathbb{V}_\gamma^x y^p)\delta y^q + M_{\gamma\ i}^{\ m}\delta W^i + N_\gamma^m dt\}) =$$

$$= -\int \frac{\partial \Gamma_{p\ q}^{\ m}}{\partial y^\ell}\frac{\partial y^\ell}{\partial x^k}(\mathbb{V}_\gamma^x y^p)\delta y^q - \int \Gamma_{p\ q}^{\ m}(\mathbb{V}_\gamma^x y^p)\delta(\frac{\partial y^q}{\partial x^k}) - \tag{24}$$

$$-\int \Gamma_{p\ q}^{\ m}(\partial_k^x \mathbb{W}_\gamma^x y^p)\delta y^q + \int \{\partial_k^x M_{\gamma\ i}^{\ m}\delta W^i + \partial_k^x N_\gamma^m dt\} \tag{25}$$

For the second and third terms in brackets in (23) we use the inductive assumption (20)

$$(23)_2 = \int \Gamma_{p\ q}^{\ m}\frac{\partial y^p}{\partial x^k}\delta(\mathbb{W}_\gamma^x y^q) + \int \Gamma_{p\ q}^{\ m}(\mathbb{W}_\gamma^x y^q)\delta(\frac{\partial y^p}{\partial x^k}) + \int \frac{\partial y^p}{\partial x^k}(\mathbb{W}_\gamma^x y^q)\delta\Gamma_{p\ q}^{\ m}(y) =$$

$$= \int \Gamma_{p\ q}^{\ m}\frac{\partial y^p}{\partial x^k}\{-\Gamma_{\ell\ s}^{\ q}(\mathbb{W}_\gamma^x y^\ell)\delta y^s + M_{\gamma\ i}^{\ m}\delta W^i + N_\gamma^q dt\} + \tag{26}$$

$$+\int \Gamma_{p\ q}^{\ m}(\mathbb{W}_\gamma^x y^q)\delta(\frac{\partial y^p}{\partial x^k}) + \int \frac{\partial y^p}{\partial x^k}(\mathbb{W}_\gamma^x y^q)\frac{\partial \Gamma_{p\ q}^{\ m}}{\partial y^\ell}\delta y^\ell \tag{27}$$

$$(23)_3 = -\sum_{s\in\gamma}\int \Gamma_{k\ s}^{\ h}(x)\{-\Gamma_{p\ q}^{\ m}(y)(\mathbb{W}_{\gamma|_{s=h}}^x y^p)\delta y^q + M_{\gamma|_{s=h}\ i}^{\ m}\delta W^i + N_{\gamma|_{s=h}}^m dt\} \tag{28}$$

Now we transform the first expression in (25) to the covariant (x)-derivative

$$(25)_1 = -\int \Gamma_{p\ q}^{\ m}(\partial_k^x \mathbb{W}_\gamma^x y^p)\delta y^q = -\int \Gamma_{p\ q}^{\ m}(\mathbb{W}_k \mathbb{W}_\gamma^x y^p)\delta y^q +$$

$$+\int \Gamma_{p\ q}^{\ m}\Gamma_{\ell\ n}^{\ p}\frac{\partial y^\ell}{\partial x^k}(\mathbb{W}_\gamma^x y^n)\delta y^q - \sum_{s\in\gamma}\int \Gamma_{p\ q}^{\ m}(y)\Gamma_{k\ s}^{\ h}(x)(\mathbb{W}_{\gamma|_{s=h}}^x y^p)\delta y^q \tag{29}$$

Expressions (24_2) and (27_1), (29_3) and (28_1) contract and the second and third terms in (25), (26) and (28) give the covariant (x)-derivatives of M and N coefficients. We write the remaining terms, redenoting indexes and gathering terms with derivatives $\partial\Gamma$ and second powers $\Gamma(y)\Gamma(y)$ of connection

$$(23) = -\int \Gamma_{p\ q}^{\ m}(\mathbb{W}_k^x \mathbb{W}_\gamma^x y^p)\delta y^q + \int \{\mathbb{W}_k^x M_{\gamma\ i}^{\ m}\delta W^i + \mathbb{W}_k^x N_\gamma^m dt\} +$$

$$+\int \frac{\partial y^\ell}{\partial x^k}(\mathbb{W}_\gamma^x y^p)\delta y^q\{\frac{\partial \Gamma_{\ell\ p}^{\ m}(y)}{\partial y^q} - \frac{\partial \Gamma_{p\ q}^{\ m}(y)}{\partial y^\ell} + \Gamma_{s\ q}^{\ m}(y)\Gamma_{\ell\ p}^{\ s}(y) - \Gamma_{\ell\ s}^{\ m}(y)\Gamma_{p\ q}^{\ s}(y)\} \tag{30}$$

But the expression in brackets $\{...\}$ gives the curvature (22), so we conclude the statement. □

3. Symmetries of variational equations and differential of norm of variation.

Similar to [2]-[7], we are going to use the symmetry of variational equations to find a set of nonlinear estimates on variations. Due to (19) the recurrence base for the definition of high order variational systems (20) is given by

$$M_{k\ i}^{\ m} = \mathbb{W}_k^x A_i^m(y_t^x), \quad N_k^m = \mathbb{W}_k^x A_0^m(y_t^x)$$

Using (18) and recurrent properties (21) we can determine the nonlinear symmetries of variational equations. Because

$$(\mathbb{W}^x)^n H(y_t^x) = \sum_{j_1+...+j_s=n,\ s=1,...,n} (\nabla^y)^s H(y_t^x)\cdot(\mathbb{W}^x)^{j_1}y...(\mathbb{W}^x)^{j_s}y \tag{31}$$

the n^{th} order variation in the l.h.s. of (20) is proportional to the n^{th} power of first variation in the r.h.s., or

$$\sqrt[n]{(\mathbb{V}^x)^n y_t^x} \sim \mathbb{V}^x y_t^x \tag{32}$$

Introduce nonlinear expression that reflects this symmetry

$$r_n(y,t) = \sum_{j=1}^{n} \mathbb{E} p_j(\rho^2(y_t^x, z)) \|(\mathbb{V}^x)^j y_t^x\|^{q/j} \tag{33}$$

and gives some nonlinear norm on the a priori smoothness of process y_t^x with respect to the initial data. Here $z \in M$ is some fixed point, $\rho(x,y)$ is geodesic distance between points x, y, norm of variation is defined by

$$\|(\mathbb{V}^x)^j y_t^x\|^2 = g_{mn}(y_t^x) \prod_{s=1}^{j} g^{i_s k_s}(x) \mathbb{V}^x_{i_1,...,i_s} y^m \mathbb{V}^x_{k_1,...,k_s} y^n \tag{34}$$

and *represents invariant* (because the variations $(\mathbb{V}^x)^j y_t^x$ form mixed tensors, one more argument in their favor!). Let us henceforth use a short notation

$$g^{\varepsilon\delta} = \prod_{s=1}^{j} g^{i_s k_s} \quad \text{for} \quad \varepsilon = \{i_1, ..., i_s\},\ \delta = \{k_1, ..., k_s\}$$

for multi-indexes ε, δ.

The next lemma prepares the differential of norm $\|(\mathbb{V}^x)^j y_t^x\|^2$, necessary for the nonlinear estimates on expression (33). First recall that by (20)-(21) the general form of variational equations looks like

$$\delta(X_\gamma^m) = -\Gamma_{p\ q}^{\ m} X_\gamma^p \delta y^q + M_{\gamma\ \alpha}^{\ m} \delta W^\alpha + N_\gamma^m dt \tag{35}$$

with coefficients $M_{\gamma\ \alpha}^{\ m}$, N_γ^m, recurrently determined in (21).

Lemma 8. *The differential of norm of variational process X_γ^m (35) has form*

$$d\|X\|^2 = g^{\gamma\varepsilon}(x) \{ g_{mn}(X_\gamma^m M_{\varepsilon\ \alpha}^{\ n} + X_\varepsilon^n M_{\gamma\ \alpha}^{\ m}) dW^\alpha +$$

$$+ g_{mn}(X_\gamma^m N_\varepsilon^n + X_\varepsilon^n N_\gamma^m + M_{\gamma\ \alpha}^{\ m} M_{\varepsilon\ \alpha}^{\ n}) dt + \frac{1}{2} g_{mn}(X_\gamma^m P_\varepsilon^n + X_\varepsilon^n P_\gamma^m) dt \ \} \tag{36}$$

Expressions P_γ^m are recurrently related by

$$P_k^m = \mathbb{V}_k^x(\nabla_{A_\alpha} A_\alpha^m) + R_{p\ \ell q}^{\ m} A_\alpha^p A_\alpha^q (\mathbb{V}_k^x y^\ell) \tag{37}$$

$$\begin{aligned} P_{\gamma\cup\{k\}}^m = {} & \mathbb{V}_k^x P_\gamma^m + 2R_{p\ \ell q}^{\ m} M_{\gamma\ \alpha}^{\ p} (\mathbb{V}_k^x y^\ell) A_\alpha^q + \\ & + (\nabla_s R_{p\ \ell q}^{\ m}) X_\gamma^p (\mathbb{V}_k^x y^\ell) A_\alpha^q A_\alpha^s + R_{p\ \ell q}^{\ m} X_\gamma^p (\mathbb{V}_k^x A_\alpha^\ell) A_\alpha^q + \\ & + R_{p\ \ell q}^{\ m} X_\gamma^p (\mathbb{V}_k^x y^\ell)(\nabla_{A_\alpha} A_\alpha) \end{aligned} \tag{38}$$

Proof. Taking the Stratonovich differential we have

$$\delta\|X\|^2 = \delta(g_{mn}(y) g^{\gamma\varepsilon}(x) X_\gamma^m X_\varepsilon^n) =$$

$$= g^{\gamma\varepsilon}(x)\,[X^m_\gamma X^n_\varepsilon \delta g_{mn}(y) + g_{mn}(X^m_\gamma \delta X^n_\varepsilon + X^n_\varepsilon \delta X^m_\gamma)]$$

Now we use $\nabla_\ell g_{mn} = 0$ in form

$$\delta g_{mn}(y) = \frac{\partial g_{mn}}{\partial y^\ell}\delta y^\ell = (g_{hn}\Gamma_{k\ m}^{\ h} + g_{mh}\Gamma_{k\ n}^{\ h})\delta y^\ell$$

to reduce the connection coefficients after substitution of δX from (35). We get

$$\delta\|X\|^2 = g^{\gamma\varepsilon}(x)\{\,g_{mn}(X^m_\gamma M_{\varepsilon\ \alpha}^{\ n} + X^n_\varepsilon M_{\gamma\ \alpha}^{\ m})\delta W^\alpha + g_{mn}(X^m_\gamma N^n_\varepsilon + X^n_\varepsilon N^m_\gamma)dt\,\} =$$

$$= g^{\gamma\varepsilon}(x)\{\,g_{mn}(X^m_\gamma M_{\varepsilon\ \alpha}^{\ n} + X^n_\varepsilon M_{\gamma\ \alpha}^{\ m})dW^\alpha +$$

$$+ g_{mn}(X^m_\gamma N^n_\varepsilon + X^n_\varepsilon N^m_\gamma)dt + \frac{1}{2}d[g_{mn}(X^m_\gamma M_{\varepsilon\ \alpha}^{\ n} + X^n_\varepsilon M_{\gamma\ \alpha}^{\ m}), W^\alpha]\,\} \tag{39}$$

where we changed from Stratonovich to Ito differentials, $[X, Y]$ denotes the quadratic variation of processes X and Y.

It remains to calculate the quadratic variation. For simplicity we omit the factor $g^{\gamma\varepsilon}(x)$

$$d[g_{mn}(X^m_\gamma M_{\varepsilon\ \alpha}^{\ n} + X^n_\varepsilon M_{\gamma\ \alpha}^{\ m}), W^\alpha] = (X^m_\gamma M_{\varepsilon\ \alpha}^{\ n} + X^n_\varepsilon M_{\gamma\ \alpha}^{\ m})d[g_{mn}(y), W^\alpha] +$$

$$+ g_{mn}(X^m_\gamma d[M_{\varepsilon\ \alpha}^{\ n}, W^\alpha] + X^n_\varepsilon d[M_{\gamma\ \alpha}^{\ m}, W^\alpha]) +$$

$$+ g_{mn}(M_{\varepsilon\ \alpha}^{\ n} d[X^m_\gamma, W^\alpha] + M_{\gamma\ \alpha}^{\ m} d[X^n_\varepsilon, W^\alpha]) = \tag{40}$$

$$= (X^m_\gamma M_{\varepsilon\ \alpha}^{\ n} + X^n_\varepsilon M_{\gamma\ \alpha}^{\ m})[g_{hn}\Gamma_{k\ m}^{\ h} + g_{mh}\Gamma_{k\ n}^{\ h}]A^\ell_\alpha dt +$$

$$+ g_{mn}(X^m_\gamma d[M_{\varepsilon\ \alpha}^{\ n}, W^\alpha] + X^n_\varepsilon d[M_{\gamma\ \alpha}^{\ m}, W^\alpha]) +$$

$$+ g_{mn} M_{\varepsilon\ \alpha}^{\ n}(-\Gamma_{p\ q}^{\ m} X^p_\gamma A^q_\alpha + M_{\gamma\ \alpha}^{\ m})dt +$$

$$+ g_{mn} M_{\gamma\ \alpha}^{\ m}(-\Gamma_{p\ q}^{\ n} X^p_\varepsilon A^q_\alpha + M_{\varepsilon\ \alpha}^{\ n})dt$$

where we used property $\nabla_\ell g_{mn} = 0$ and substituted differentials (35). Contracting the connections we get finally

$$d[g_{mn}(X^m_\gamma M_{\varepsilon\ \alpha}^{\ n} + X^n_\varepsilon M_{\gamma\ \alpha}^{\ m}), W^\alpha] = 2g_{mn} M_{\gamma\ \alpha}^{\ m} M_{\varepsilon\ \alpha}^{\ n} dt +$$

$$+ g_{mn} X^n_\varepsilon (\,d[M_{\gamma\ \alpha}^{\ m}, W^\alpha] + \Gamma_{p\ q}^{\ m} M_{\gamma\ \alpha}^{\ p} A^q_\alpha dt\,) + \tag{41}$$

$$+ g_{mn} X^m_\gamma (\,d[M_{\varepsilon\ \alpha}^{\ n}, W^\alpha] + \Gamma_{p\ q}^{\ n} M_{\varepsilon\ \alpha}^{\ p} A^q_\alpha dt\,)$$

Introduce notation for the terms in round brackets in (41)

$$P^m_\gamma dt = d[M_{\gamma\ \alpha}^{\ m}, W^\alpha] + \Gamma_{p\ q}^{\ m} M_{\gamma\ \alpha}^{\ p} A^q_\alpha dt \tag{42}$$

and find the recurrent way to calculate expression P^m_γ.

We use (73) to obtain the necessary relations

$$P^m_{\gamma\cup\{k\}}dt = d[M_{\gamma\cup\{k\}\ \alpha}^{\ m}, W^\alpha] + \Gamma_{p\ q}^{\ m} M_{\gamma\cup\{k\}\ \alpha}^{\ p} A^q_\alpha dt =$$

$$= d[\mathbb{V}^x_k M_{\gamma\ \alpha}^{\ m}, W^\alpha] + \Gamma_{p\ q}^{\ m}(\mathbb{V}^x_k M_{\gamma\ \alpha}^{\ p})A^q_\alpha dt + \tag{43}$$

$$+ d[R_{p\ \ell q}^{\ m} X^p_\gamma (\mathbb{V}^x_k y^\ell) A^q_\alpha, W^\alpha] + \Gamma_{p\ q}^{\ m}(R_{i\ \ell j}^{\ p} X^i_\gamma (\mathbb{V}^x_k y^\ell) A^j_\alpha) A^q_\alpha dt \tag{44}$$

The last line (44) appears only for $\gamma \neq \emptyset$, therefore we first calculate line (43),

1. *Transformation of line (43).* First we substitute the definition of covariant (x)-derivative

$$(43) = d[\partial^x_k M_{\gamma}{}^{m}{}_{\alpha} + \Gamma_{p}{}^{m}{}_{q} M_{\gamma}{}^{p}{}_{\alpha} \mathbb{V}^x_k y^q - \sum_{s\in\gamma} \Gamma_{k}{}^{h}{}_{s}(x) M_{\gamma|_{s=h}}{}^{m}{}_{\alpha}, W^\alpha]+ \quad (45)$$

$$+\Gamma_{p}{}^{m}{}_{q}(\partial^x_k M_{\gamma}{}^{m}{}_{\alpha} + \Gamma_{i}{}^{p}{}_{j} M_{\gamma}{}^{m}{}_{\alpha}(\mathbb{V}^x_k y^j) - \sum_{s\in\gamma} \Gamma_{k}{}^{h}{}_{s}(x) M_{\gamma|_{s=h}}{}^{p}{}_{\alpha}) A^q_\alpha dt \quad (46)$$

For the first term in (45) we take the differentiation outside and form covariant (x)-derivative by adding and subtracting necessary terms

$$d[\partial^x_k M_{\gamma}{}^{m}{}_{\alpha}, W^\alpha] = \partial^x_k(d[M_{\gamma}{}^{m}{}_{\alpha}, W^\alpha]) =$$

$$= \mathbb{V}^x_k(d[M_{\gamma}{}^{m}{}_{\alpha}, W^\alpha]) - \Gamma_{p}{}^{m}{}_{q} d[M_{\gamma}{}^{p}{}_{\alpha}, W^\alpha]\mathbb{V}^x_k y^q + \sum_{s\in\gamma} \Gamma_{k}{}^{h}{}_{s}(x) d[M_{\gamma|_{s=h}}{}^{m}{}_{\alpha}, W^\alpha] \quad (47)$$

The second and third terms in (45) are calculated directly

$$(45)_{2+3} = M_{\gamma}{}^{p}{}_{\alpha}(\mathbb{V}_k y^q) d[\Gamma_{p}{}^{m}{}_{q}, W^\alpha] + \Gamma_{p}{}^{m}{}_{q} M_{\gamma}{}^{p}{}_{\alpha} d[\mathbb{V}^x_k y^q, W^\alpha]+$$

$$+\Gamma_{p}{}^{m}{}_{q}(\mathbb{V}^x_k y^q) d[M_{\gamma}{}^{p}{}_{\alpha}, W^\alpha] - \sum_{s\in\gamma} \Gamma_{k}{}^{h}{}_{s}(x) d[M_{\gamma|_{s=h}}{}^{m}{}_{\alpha}, W^\alpha] \quad (48)$$

Noting that two terms in (48) are compensated by two terms on (47) and substituting the differential of $\mathbb{V}^x_k y^q$ from (35) we have for line (45)

$$(45) = (47) + (48) = \mathbb{V}^x_k(d[M_{\gamma}{}^{m}{}_{\alpha}, W^\alpha])+ \quad (49)$$

$$+M_{\gamma}{}^{p}{}_{\alpha}(\mathbb{V}^x_k y^q)\frac{\partial \Gamma_{p}{}^{m}{}_{q}}{\partial y^\ell} A^\ell_\alpha dt + \Gamma_{p}{}^{m}{}_{q} M_{\gamma}{}^{p}{}_{\alpha}(-\Gamma_{i}{}^{q}{}_{j}(\mathbb{V}^x_k y^j) A^i_\alpha + M_{k}{}^{q}{}_{\alpha}) dt \quad (50)$$

For the first term in (46) we take the derivative ∂^x_k outside of all terms and form $\mathbb{V}^x$ derivative by adding and subtracting necessary terms

$$(46)_1 = \partial^x_k(\Gamma_{p}{}^{m}{}_{q} M_{\gamma}{}^{p}{}_{\alpha} A^q_\alpha) dt - [\partial^x_k \Gamma_{p}{}^{m}{}_{q}(y)] M_{\gamma}{}^{p}{}_{\alpha} A^q_\alpha dt-$$

$$-\Gamma_{p}{}^{m}{}_{q} M_{\gamma}{}^{p}{}_{\alpha}(\partial^x_k A^q_\alpha) dt = \mathbb{V}^x_k(\Gamma_{p}{}^{m}{}_{q} M_{\gamma}{}^{p}{}_{\alpha} A^q_\alpha) dt-$$

$$-\Gamma_{h}{}^{m}{}_{\ell}(\Gamma_{p}{}^{h}{}_{q} M_{\gamma}{}^{p}{}_{\alpha} A^q_\alpha)\mathbb{V}^x_k y^\ell dt + \sum_{s\in\gamma} \Gamma_{k}{}^{h}{}_{s}(x)(\Gamma_{p}{}^{m}{}_{q} M_{\gamma|_{s=h}}{}^{p}{}_{\alpha} A^q_\alpha) dt- \quad (51)$$

$$-\frac{\partial \Gamma_{p}{}^{m}{}_{q}}{\partial y^\ell}(\mathbb{V}^x_k y^\ell) M_{\gamma}{}^{p}{}_{\alpha} A^q_\alpha dt - \Gamma_{p}{}^{m}{}_{q} M_{\gamma}{}^{p}{}_{\alpha}(\partial^x_k A^q_\alpha) dt \quad (52)$$

Note first that term $(46)_3$ is compensated by term $(51)_2$. Because by (73)

$$M_{k}{}^{q}{}_{\alpha} = \mathbb{V}^x_k A^q_\alpha(y) = \partial^x_k A^q_\alpha(y) + \Gamma_{i}{}^{q}{}_{j} A^i_\alpha(\mathbb{V}^x_k y^j)$$

we have that $(50)_2$+$(50)_3$ compensate $(52)_2$. Collecting all remaining terms we have

$$(43) = (45) + (46) = \mathbb{V}^x_k P^m_\gamma dt+$$

$$+\{\frac{\partial \Gamma_{p}{}^{m}{}_{\ell}}{\partial y^q} - \frac{\partial \Gamma_{p}{}^{m}{}_{q}}{\partial y^\ell} + \Gamma_{h}{}^{m}{}_{q}\Gamma_{p}{}^{h}{}_{\ell} - \Gamma_{h}{}^{m}{}_{\ell}\Gamma_{p}{}^{h}{}_{q}\} M_{\gamma}{}^{p}{}_{\alpha}(\mathbb{V}^x_k y^\ell) A^q_\alpha dt = \quad (53)$$

$$= \{\mathbb{V}^x_k P^m_\gamma + R_p{}^m{}_{\ell q} M_\gamma{}^p{}_\alpha (\mathbb{V}^x_k y^\ell) A^q_\alpha\} dt \tag{54}$$

where terms in line (53) appear from $(50)_1$, $(52)_1$, $(46)_2$ and $(51)_1$.

Because for recurrence base $\gamma = \emptyset$ terms in (44) do not appear, it remains to find $P^m_\emptyset$ to finish recurrence base (37). By definitions (72), (42) we have

$$P^m_\emptyset dt = d[A^m_\alpha(y), W^\alpha] + \Gamma_\ell{}^m{}_h A^\ell_\alpha A^h_\alpha dt =$$

$$= [\frac{\partial A^m_\alpha}{\partial y^\ell} + \Gamma_\ell{}^m{}_h A^h_\alpha] A^\ell_\alpha dt = (\nabla_\ell A^m_\alpha) \cdot A^\ell_\alpha dt = (\nabla_{A_\alpha} A^m_\alpha) dt \tag{55}$$

i.e. from (54) and (55) follows (37).

2. *Transformation of line (44).* Taking the first term in (44) we obtain

$$(44)_1 = d[R_p{}^m{}_{\ell q} X^p_\gamma (\mathbb{V}^x_k y^\ell) A^q_\alpha, W^\alpha] =$$

$$= \frac{\partial R_p{}^m{}_{\ell q}}{\partial y^s} X^p_\gamma (\mathbb{V}^x_k y^\ell) A^q_\alpha A^s_\alpha dt + R_p{}^m{}_{\ell q} (\mathbb{V}^x_k y^\ell) A^q_\alpha d[X^p_\gamma, W^\alpha] + \tag{56}$$

$$+ R_p{}^m{}_{\ell q} X^p_\gamma A^q_\alpha d[\mathbb{V}^x_k y^\ell, W^\alpha] + R_p{}^m{}_{\ell q} X^p_\gamma (\mathbb{V}^x_k y^\ell) \frac{\partial A^q_\alpha}{\partial y^s} A^s_\alpha dt = \tag{57}$$

$$= \frac{\partial R_p{}^m{}_{\ell q}}{\partial y^s} X^p_\gamma (\mathbb{V}^x_k y^\ell) A^q_\alpha A^s_\alpha dt + \tag{58}$$

$$+ R_p{}^m{}_{\ell q} (\mathbb{V}^x_k y^\ell) A^q_\alpha [-\Gamma_i{}^p{}_j X^i_\gamma A^j_\alpha + M_\gamma{}^p{}_\alpha] dt + \tag{59}$$

$$+ R_p{}^m{}_{\ell q} X^p_\gamma A^q_\alpha [-\Gamma_i{}^\ell{}_j (\mathbb{V}^x_k y^i) A^j_\alpha + M_k{}^\ell{}_\alpha] dt + \tag{60}$$

$$+ R_p{}^m{}_{\ell q} X^p_\gamma (\mathbb{V}^x_k y^\ell) [\nabla_s A^q_\alpha - \Gamma_s{}^q{}_h A^h_\alpha] A^s_\alpha dt \tag{61}$$

where we transformed to the $\mathbb{V}^x$ derivatives, by changing $(56)_2$ to (59), term $(57)_1$ to (60). We also transformed the partial derivative of A_α in $(57)_2$ to the covariant one.

Collecting together above expressions and $(44)_2$ we have

$$(44) = \{\frac{\partial R_p{}^m{}_{\ell q}}{\partial y^s} - \Gamma_p{}^h{}_s R_h{}^m{}_{\ell q} + \Gamma_h{}^m{}_s R_p{}^h{}_{\ell q} - \tag{62}$$

$$- \Gamma_\ell{}^h{}_s R_p{}^m{}_{hq} - \Gamma_q{}^h{}_s R_p{}^m{}_{\ell h}\} X^p_\gamma (\mathbb{V}^x_k y^\ell) A^q_\alpha A^s_\alpha dt + \tag{63}$$

$$+ R_p{}^m{}_{\ell q} M_\gamma{}^p{}_\alpha (\mathbb{V}^x_k y^\ell) A^q_\alpha dt + R_p{}^m{}_{\ell q} X^p_\gamma (\mathbb{V}^x_k A^\ell_\alpha) A^q_\alpha dt + \tag{64}$$

$$+ R_p{}^m{}_{\ell q} X^p_\gamma (\mathbb{V}^x_k y^\ell) (\nabla_s A^q_\alpha) A^s_\alpha dt \tag{65}$$

Here $(64)_1 = (59)_2$, $(64)_2 = (60)_2$ and $(65) = (61)_1$. Terms in brackets in (62)-(63) appear correspondingly from (58), $(59)_1$, $(44)_2$, $(60)_1$, $(61)_2$ and form the covariant derivative of the curvature tensor.

We get finally

$$(44)/dt = (\nabla_s R_p{}^m{}_{\ell q}) X^p_\gamma (\mathbb{V}^x_k y^\ell) A^q_\alpha A^s_\alpha + R_p{}^m{}_{\ell q} M_\gamma{}^p{}_\alpha (\mathbb{V}^x_k y^\ell) A^q_\alpha +$$

$$+ R_p{}^m{}_{\ell q} X^p_\gamma (\mathbb{V}^x_k A^\ell_\alpha) A^q_\alpha + R_p{}^m{}_{\ell q} X^p_\gamma (\mathbb{V}^x_k y^\ell) (\nabla_{A_\alpha} A^q_\alpha)$$

that together with (54) leads to (38). □

4. Nonlinear estimate on new type variations.

Using Lemma 8 we can find nonlinear estimates on variations.

The following theorem provides necessary conditions for quasi-contractive estimate on r_n, generalizing results of [2]-[7] to the noncompact manifold setting. Introduce notation

$$\widetilde{A_0} = A_0 + \frac{1}{2}\sum_{\alpha=1}^{d} \nabla_{A_\alpha} A_\alpha \tag{66}$$

THEOREM 9. *Suppose that the following conditions hold*

- **dissipativity**: *$\exists z \in M$ such that $\forall C \in I\!R^1\ \exists K_C \in I\!R^1$ such that $\forall x \in M$*

$$< \widetilde{A_0}(x), \nabla^x \rho^2(x,z) > +C\sum_{\alpha=1}^{d} \|A_\alpha(x)\|^2 \le K_C(1+\rho^2(x,z)) \tag{67}$$

- **differential coercitivity**: *$\forall C, C' \in I\!R^1\ \exists K_C \in I\!R^1$ such that $\forall x, y \in M$*

$$< \nabla\widetilde{A_0}[h], h > +C\sum_{\alpha=1}^{d} |\nabla A_\alpha[h]|^2 + C'\sum_{\alpha=1}^{d} < R(A_\alpha, h)A_\alpha, h > \le K_C\|h\|^2 \tag{68}$$

Notation $\nabla A[h] = h^\ell \nabla_\ell A$ means covariant directional derivative, $< \cdot, \cdot >$ and $|\cdot|$ corresponding Riemannian scalar product and norm, and

$$[R(A,B)C]^m = R_{i\ jk}^{\ m} A^i B^j C^k$$

denotes curvature operator

- **nonlinear behaviour of coefficients and curvature**: *for any n there are constants $\mathbf{k}_\bullet$ such that for all $j = 1, .., n$ and $\forall x \in M$*

$$\|(\nabla)^j \widetilde{A_0}(x)\| \le (1+\rho(x,z))^{\mathbf{k}_0}$$

$$\|(\nabla)^j A_\alpha(x)\| \le (1+\rho(x,z))^{\mathbf{k}_\alpha} \tag{69}$$

$$\|(\nabla)^j R(x)\| \le (1+\rho(x,z))^{\mathbf{k}_R}$$

Then there is some $\mathbf{k} = \mathbf{k}(\mathbf{k}_0, \mathbf{k}_\alpha, \mathbf{k}_R)$ such that if monotone polynomials $p_j \ge 1$ in (33) are hierarchied by

$$\forall j_1 + j_s = i \le n \quad [p_i(\cdot)]^i (1+|\cdot|^2)^{\mathbf{k}_q} \le [p_{j_1}(\cdot)]^{j_1} ... [p_{j_s}(\cdot)]^{j_s} \tag{70}$$

then the nonlinear estimate on variations holds

$$\exists K_{\mathbf{k}} \quad \forall t \ge 0 \quad r_n(y,t) \le e^{K_{\mathbf{k}} t} r_n(y,0) \tag{71}$$

REMARK 10. Conditions (67)-(68) generalize the classical Krylov - Rosovskii - Pardoux conditions [23, 19] from the linear base space to manifold, here arises the additional term with curvature.

Proof. Step 1. Structure of the differential of norm $\|(\mathbb{W}^x)^j y_t^x\|^2$.

To simplify further notations, let us introduce an additional process, that formally corresponds to the index $\gamma = \emptyset$ in (35)

$$\delta X_\emptyset^m = -\Gamma_{p\ q}^{\ m} X_\emptyset^p \delta y^q + A_\alpha^m \delta W^\alpha + A_0^m dt$$

Then the relations of coefficients M, N for the processes X_γ^m could be written in the following form

1. recurrence base:
$$M_{\emptyset\ \alpha}^{\ m} = A_\alpha^m(y_t^x), \quad N_\emptyset^m = A_0^m(y_t^x) \tag{72}$$

2. recurrence step by (19) for $\gamma = \emptyset$ and (21) for $\gamma \neq \emptyset$
$$M_{\gamma\cup\{k\}\ \alpha}^{\ m} = \begin{cases} \mathbb{W}_k^x M_{\emptyset\ \alpha}^{\ m}, & \text{for } \gamma = \emptyset \\ \mathbb{W}_k^x M_{\gamma\ \alpha}^{\ m} + R_{p\ \ell q}^{\ m} X_\gamma^p (\mathbb{W}_k^x y^\ell) A_\alpha^q, & \text{for } \gamma \neq \emptyset \end{cases} \tag{73}$$

$$N_{\gamma\cup\{k\}}^m = \begin{cases} \mathbb{W}_k^x N_\emptyset^m, & \text{for } \gamma = \emptyset \\ \mathbb{W}_k^x N_\gamma^m + R_{p\ \ell q}^{\ m} X_\gamma^p (\mathbb{W}_k^x y^\ell) A_0^q, & \text{for } \gamma \neq \emptyset \end{cases} \tag{74}$$

Therefore, due to (18), (31) and (72)-(74), we have that coefficients of the high order variational equations have form

$$M_{\gamma\ \alpha}^{\ m} = \nabla_\ell^y A_\alpha^m [\mathbb{W}_\gamma^x y^\ell] + \sum_{\beta_1\cup\ldots\cup\beta_s=\gamma,\ s\geq 2} K'_{\beta_1,\ldots,\beta_s}(\mathbb{W}_{\beta_1}^x y, \ldots, \mathbb{W}_{\beta_s}^x y) \tag{75}$$

$$N_\gamma^{\ m} = \nabla_\ell^y A_\alpha^0 [\mathbb{W}_\gamma^x y^\ell] + \sum_{\beta_1\cup\ldots\cup\beta_s=\gamma,\ s\geq 2} K''_{\beta_1,\ldots,\beta_s}(\mathbb{W}_{\beta_1}^x y, \ldots, \mathbb{W}_{\beta_s}^x y)$$

with coefficients K', K'', depending on A_0, A_α, R and their covariant derivatives. Moreover, the dependence of $K_{\beta_1,\ldots,\beta_s}(\mathbb{W}_{\beta_1}^x y, \ldots, \mathbb{W}_{\beta_s}^x y)$ on lower order variations $\mathbb{W}_\beta^x y$ manifests symmetries (32).

Let now $i = 1$ and $X_k^m = \mathbb{W}_l^x y^m$, then by (37) and (18)

$$P_k^m = \mathbb{W}_k^x (\nabla_{A_\alpha} A_\alpha^m(y)) + R_{p\ \ell q}^{\ m} A_\alpha^p A_\alpha^q \mathbb{W}_k^x y^\ell =$$

$$= \nabla_\ell^y \nabla_{A_\alpha} A_\alpha^m \cdot \mathbb{W}_k^x y^\ell + R(A_\alpha, \mathbb{W}_k^x y) A_\alpha$$

Therefore, because in (38) $P_{\gamma\cup\{k\}}^m = \mathbb{W}_k^x P_\gamma^m + \ldots$ the high order coefficients permit representation

$$P_\gamma^m = \nabla_\ell \nabla_{A_\alpha} A_\alpha^m \cdot \mathbb{W}_\gamma^x y^\ell + R(A_\alpha, \mathbb{W}_\gamma^x y) A_\alpha + \sum_{\beta_1\cup..\cup\beta_s=\gamma,\ s\geq 2} K_{\beta_1,\ldots,\beta_s}(\mathbb{W}_{\beta_1}^x y, \ldots, \mathbb{W}_{\beta_s}^x y)$$

with symmetries (32) in $K_{\beta_1,\ldots,\beta_s}$ terms, depending on A_0, A_α, R and their covariant derivatives.

Therefore from (36) the differential of norm admits representation

$$d\|(\mathbb{W}^x)^i y_t^x\|^2 = 2 < (\mathbb{W})^i y, \nabla_\ell^y A_\alpha[(\mathbb{W})^i y^\ell] > dW^\alpha + \{\, 2 < (\mathbb{W})^i y, \nabla_\ell^y \widetilde{A_0}(\mathbb{W})^i y^\ell] > +$$
$$+ \sum_{\alpha=1}^{d} \|\nabla A_\alpha[(\mathbb{W})^i y]\,\|^2 + \sum_{\alpha=1}^{d} < R(A_\alpha, (\mathbb{W})^i y)A_\alpha, (\mathbb{W})^i y > \}dt+$$
$$+ \sum_{j_1+\ldots+j_s=i,\ s\geq 2} < (\mathbb{W}^x)^i y, \{K^1_{j_1,\ldots,j_s,\alpha}((\mathbb{W})^{j_1}y, \ldots, (\mathbb{W})^{j_s}y)dW^\alpha + \tag{76}$$
$$+K^2_{j_1,\ldots,j_s}((\mathbb{W})^{j_1}y, \ldots, (\mathbb{W})^{j_s}y)dt\} >$$

i.e. the *coercitivity condition* arises in the principal part. Like before the coefficients K^1, K^2 depend on covariant derivatives of A_0, A_α, R and display symmetry (32).

Step 2. Separation of the principal part. Writing the differential of one terms in nonlinear expression (33) we have by Ito formula (temporarily $2q = m/i$, $p = p_i$)

$$h(t) = \mathbb{E}p(\rho^2(y_t^x, z))\|(\mathbb{W}^x)^i y_t^x\|^{2q} = h(0) + \mathbb{E}\int_0^t\{p(\rho^2(y_s^x, z))\, d\|(\mathbb{W}^x)^i y_s^x\|^{2q} +$$
$$+\|(\mathbb{W}^x)^i y_s^x\|^{2q} dp(\rho^2(y_s^x, z)) + \tfrac{1}{2}d[p(\rho^2(y_s^x, z)), \|(\mathbb{W}^x)^i y_s^x\|^{2q}]\,\} =$$
$$= h(0) + \int_0^t \mathbb{E}\{p(\rho^2(y_s^x, z))(2q\|(\mathbb{W}^x)^i y_s^x\|^{2(q-1)} d\|(\mathbb{W}^x)^i y_s^x\|^2 + \tag{77}$$
$$+q(2q-2)\|(\mathbb{W}^x)^i y_s^x\|^{2(q-2)} d[\|(\mathbb{W}^x)^i y_s^x\|^2, \|(\mathbb{W}^x)^i y_s^x\|^2] +$$
$$+\|(\mathbb{W}^x)^i y_s^x\|^{2q}(p'(\rho^2(y_s^x, z))d\rho^2(y_s^x, z) + \frac{1}{2}p''(\rho^2(y_s^x, z))d[\rho^2(y_s^x, z), \rho^2(y_s^x, z)]) + \tag{78}$$
$$+\frac{1}{2}p'(\rho^2(y_s^x, z))\|(\mathbb{W}^x)^i y_s^x\|^{2(q-1)} d[\rho^2(y_s^x, z), \|(\mathbb{W}^x)^i y_s^x\|^2]\,\} \tag{79}$$

By (76) terms in (77) give the coercitivity condition (68) in principal part with some constants and additional terms with lower order variations

$$(77) \leq K\mathbb{E}\int_0^t p(\rho^2(y_t^x, z))\|(\mathbb{W})^i y_t^x\|^{2(q-1)}\{\text{coercitivity}\}_{C,C'}((\mathbb{W})^i y_t^x, (\mathbb{W})^i y_t^x)dt+$$
$$+ \sum_{j_1+\ldots+j_s=i,\ s\geq 2} \mathbb{E}\int_0^t p(\rho^2(y_t^x, z))\|(\mathbb{W})^i y_t^x\|^{2(q-1)} < (\mathbb{W})^i y, K_{j_1,\ldots,j_s}((\mathbb{W})^{j_1}y, \ldots, (\mathbb{W})^{j_s}y) > dt \tag{80}$$

with coefficients K as before.

Term in (78) is transformed by monotonicity and polynomiality of p ($\exists C:\ p''(u)u \leq Cp'(u)$), index [1] means that the corresponding operators act on the first coordinate

$$\int_0^1 \mathbb{E}\|(\mathbb{W})^i y\|^{2q}\{p'(\rho^2(y,z))d\rho^2(y,z) + \frac{1}{2}p''(\rho^2(y,z))d[\rho^2(y,z), \rho^2(y,z)] =$$

$$= \int_0^1 \mathbb{E}\|(\mathbb{W})^i y\|^{2q}\{p'(\rho^2(y,z))L^1\rho^2(y,z) + \frac{1}{2}p''(\rho^2(y,z))\rho^2(y,z)\frac{1}{\rho^2(y,z)}\sum_{\alpha=1}^{d}(A^1_\alpha\rho^2(y,z))^2\}dt \leq$$

$$\le \int_0^t \mathbb{E}\|(\mathbb{V})^i y\|^{2q} p'(\rho^2(y,z))\{L^1\rho^2(y,z) + \frac{C}{\rho^2(y,z)}\sum_{\alpha=1}^d (A^1_\alpha \rho^2(y,z)\,)^2\}dt \qquad (81)$$

after that work results of [6] about the optimal estimates on general second order operators on metric functions.

THEOREM 11. *([6]). Suppose that the generalized dissipativity and coercitivity conditions (67)-(68) hold.*

Then there is constant K such that

$$L^1\rho^2(x,y) \le K(1+\rho^2(x,y)) \qquad (82)$$

Moreover $\forall\, C\ \exists K_C$ such that

$$L^1\rho^2(x,y) + C\sum_{\alpha=1}^d \frac{(A^1_\alpha\rho^2(x,y))^2}{\rho^2(x,y)} \le K_C(1+\rho^2(x,y)) \qquad (83)$$

By (75) terms (79) are estimated by

$$|p'(\rho^2)\,\|(\mathbb{V})^i y\|^{2(q-1)} d[\rho^2, \|(\mathbb{V})^i y^2]\,| \le$$

$$\le p'(\rho^2)\|(\mathbb{V})^i y\|^{2q}\sum_{\alpha=1}^d \frac{(A^1_\alpha \rho^2))^2}{\rho^2} +$$

$$+p'(\rho^2)\rho^2\|(\mathbb{V})^i y\|^{2(q-1)}\|\nabla A_\alpha[(\mathbb{V})^i y] + \sum_{j_1+\ldots+j_s,\ s\ge 2} K'_{j_1,\ldots,j_s}((\mathbb{V})^{j_1}y,\ldots,(\mathbb{V})^{j_s}y)\|^2 \qquad (84)$$

The first term is added to (81), after that (83) works. The second term is combined with terms in (77), (80), leading to the coercitivity condition with modified constants.

Therefore, after the application of coercitivity and dissipativity assumptions (67)-(68), we come to estimate

$$h(t) = \mathbb{E}p(\rho^2(y^x_t,z))\|(\mathbb{V})^i y^x_t\|^{2q} \le h(0) + C\int_0^t h(t)dt +$$

$$+ \sum_{j_1+\ldots+j_s,\ s\ge 2}\int_0^t \mathbb{E}p(\rho^2(y,z))\|(\mathbb{V})^i y\|^{2(q-1)} K'_{i;j_1,\ldots,j_s}((\mathbb{V})^i y; (\mathbb{V})^{j_1}y,\ldots,(\mathbb{V})^{j_s}y)dt$$

where coefficient K' reflects symmetry (32) and can depend quadratically on lower order variations for the case of (84).

Step 3. Estimation of the rest terms with the use of nonlinear symmetry (32).

Finally, in a similar to [2]-[7] way, we may apply nonlinear behaviour (69), hierarchy (70) and symmetry (32) to deal with the rest terms. By inequality $|x^{q-1}y| \le |x|^q/q + (q-1)|y|^q/q$ we have for $a = 1,2$

$$\mathbb{E}p(\rho^2)\,\|(\mathbb{V})^i y\|^{2(q-1)} K_{i;j_1,\ldots,j_s}((\mathbb{V})^i y; (\mathbb{V})^{j_1}y,\ldots,(\mathbb{V})^{j_s}y) \le$$

$$\le \mathbb{E}p(\rho^2)(1+\rho^2)^{\mathbf{k}}\|(\mathbb{V})^i y\|^{2q-a}\|(\mathbb{V})^{j_1}y\|^a \ldots \|(\mathbb{V})^{j_s}y\|^a \le$$

$$\leq C\mathbb{E}p\|(\mathbb{W})^i y\|^{2q} + C'\mathbb{E}p(\rho^2)(1+\rho^2)^{2q\mathbf{k}}\|(\mathbb{W})^{j_1}y\|^{2q}...\|(\mathbb{W})^{j_s}y\|^{2q}$$

The first term equals $h(s)$, for the last we recall that $2q = m/i$ (33), so there is representation

$$\|x_{j_1}\|^{m/i}...\|x_{j_s}\|^{m/i} = (\|x_{j_1}\|^{m/j_1})^{j_1/i}...(\|x_{j_s}\|^{m/j_s})^{j_s/i}$$

Then the nonlinear hierarchies of polynomials (70) give

$$p_i(\rho^2)(1+\rho^2)^{\mathbf{k}m/i}\|(\mathbb{W})^{j_1}y\|^{m/i}...\|(\mathbb{W})^{j_s}y\|^{m/i} \leq$$

$$\leq (p_{j_1}(\rho^2)\,\|(\mathbb{W})^{j_1}y\|^{m/j_1})^{j_1/i}...(p_{j_s}(\rho^2)\,\|(\mathbb{W})^{j_s}y\|^{m/j_s})^{j_s/i} \leq$$

$$\leq \frac{j_1}{i}p_{j_1}(\rho^2)\,\|(\mathbb{W})^{j_1}y\|^{m/j_1} + ... + \frac{j_s}{i}p_{j_s}(\rho^2)\,\|(\mathbb{W})^{j_s}y\|^{m/j_s}$$

i.e. the differential of each term in (33) is estimated by terms of (33) itself

$$h_i(t) = \mathbb{E}p_i(\rho^2)\,\|(\mathbb{W})^i y\|^{q/i} \leq h_i(0) + const\int_0^t r_n(y,s)ds$$

Application of Gronwall-Bellmann inequality leads to (71). □

Similar to [2]-[7] under conditions on coefficients of diffusion equation, that lead to the a priori estimate on variations, we also have C^∞ regularity of process y_t^x and regular properties of diffusion semigroup. The next theorem announces the result.

THEOREM 12. *Under conditions (67)-(69) process y_t^x is C^∞ differentiable with respect to the initial data. Its variations $(\mathbb{W}^x)y_t^x$ represent strong solutions to variational systems (20)-(21).*

Moreover, there is $\mathbf{k}$ *such that for a family $q_0, q_1, ..., q_n \geq 1$ of monotone functions of polynomial behaviour, that fulfill*

$$\forall\, i\geq 1 \quad q_i(b)(1+b)^{\mathbf{k}} \leq q_{i+1}(b)\ \forall\, b\geq 0 \tag{85}$$

the space $C_{\vec{q}}^n(M)$, consisting of n-times continuously covariantly differentiable functions with a finite norm

$$\|f\|_{C_{\vec{q}}^n(M)} = \max_{i=0,...,n}\sup_{x\in M}\frac{\|(\nabla^x)^i f(x)\|}{q_i(\rho^2(x,z))}, \tag{86}$$

is preserved under the action of semigroup

$$\forall\, t\geq 0\ \ P_t : C_{\vec{q}}^n(M) \to C_{\vec{q}}^n(M)$$

Furthermore, there are constants K, M such that the quasi-contractive estimate holds

$$\forall\, f\in C_{\vec{q}}^n(M) \qquad \|P_t f\|_{C_{\vec{q}}^n(M)} \leq Ke^{Mt}\|f\|_{C_{\vec{q}}^n(M)} \tag{87}$$

Proof will appear in [8].

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